Hodgkin-Huxley Model
and
FitzHugh-Nagumo Model
Nervous System

- Signals are propagated from nerve cell to nerve cell (*neuron*) via electro-chemical mechanisms
- ~100 billion neurons in a person
- Hodgkin and Huxley experimented on squids and discovered how the signal is produced within the neuron
- H.-H. model was published in *Jour. of Physiology* (1952)
- H.-H. were awarded 1963 Nobel Prize
- FitzHugh-Nagumo model is a simplification
When the axon is excited, $V$ spikes because sodium $Na^+$ and potassium $K^+$ ions flow through the membrane.

Axon membrane potential difference

$$V = V_i - V_e$$
Nernst Potential

$V_{Na}$, $V_{K}$ and $V_{r}$

Ion flow due to electrical signal

Traveling wave

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Since the membrane separates charge, it is modeled as a capacitor with capacitance $C$. Ion channels are resistors.

$$\frac{1}{R} = g = \text{conductance}$$

V_{K} \quad V_{Na} \quad V_{r}

$C$

$g_{K}$ $g_{Na}$ $g_{r}$

$i_{C} = C \frac{dV}{dt}$

$i_{Na} = g_{Na} (V - V_{Na})$

$i_{K} = g_{K} (V - V_{K})$

$i_{r} = g_{r} (V - V_{r})$
Since the sum of the currents is 0, it follows that

\[ C \frac{dV}{dt} = -g_{Na}(V - V_{Na}) - g_K(V - V_K) - g_r(V - V_r) + I_{ap} \]

where \( I_{ap} \) is applied current. If ion conductances are constants then group constants to obtain 1st order, linear eq

\[ C \frac{dV}{dt} = -g(V - V^*) + I_{ap} \]

Solving gives

\[ V(t) \rightarrow V^* + \frac{I_{ap}}{g} \]
Experiments showed that $g_{Na}$ and $g_{K}$ varied with time and V. After stimulus, Na responds much more rapidly than K.
**Hodgkin-Huxley System**

Four state variables are used:

\[ v(t) = V(t) - V_{eq} \] is membrane potential, 

\[ m(t) \] is Na activation, 

\[ n(t) \] is K activation and 

\[ h(t) \] is Na inactivation.

In terms of these variables \( g_K = g_K n^4 \) and \( g_{Na} = g_{Na} m^3 h. \)

The resting potential \( V_{eq} \approx -70 \text{mV}. \) Voltage clamp experiments determined \( g_K \) and \( n \) as functions of \( t \) and hence the parameter dependences on \( v \) in the differential eq. for \( n(t) \). Likewise for \( m(t) \) and \( h(t) \).
Hodgkin-Huxley System

\[
C \frac{d\nu}{dt} = -g_{Na} m^3 h (\nu - V_{Na}) - g_K n^4 (\nu - V_K) - g_r (\nu - V_r) + I_{ap}
\]

\[
\frac{dm}{dt} = \alpha_m(\nu)(1 - m) - \beta_m(\nu)m
\]

\[
\frac{dn}{dt} = \alpha_n(\nu)(1 - n) - \beta_n(\nu)n
\]

\[
\frac{dh}{dt} = \alpha_h(\nu)(1 - h) - \beta_h(\nu)h
\]
$I_{ap} = 8, v(t)$

$I_{ap} = 7, v(t)$
Fast-Slow Dynamics

\[ \rho_m(v) \frac{dm}{dt} = m_\infty(v) - m. \]

\( \rho_m(v) \) is much smaller than \( \rho_n(v) \) and \( \rho_h(v) \). An increase in \( v \) results in an increase in \( m_\infty(v) \) and a large \( \frac{dm}{dt} \).

Hence Na activates more rapidly than K in response to a change in \( v \).

\( v, m \) are on a fast time scale and \( n, h \) are slow.
FitzHugh-Nagumo System

\[ \varepsilon \frac{dv}{dt} = f(v) - w + I \quad \text{and} \quad \frac{dw}{dt} = v - 0.5w \]

\[ I \] represents applied current, \( \varepsilon \) is small and \( f(v) \) is a cubic nonlinearity. Observe that in the \((v,w)\) phase plane which is small unless the solution is near \( f(v) - w + I = 0 \). Thus the \textit{slow manifold} is the cubic \( w = f(v) + I \) which is the \textit{nullcline} of the fast variable \( v \). And \( w \) is the slow variable with \textit{nullcline} \( w = 2v \).
Take $f(v) = v(1-v)(v-a)$.

Stable rest state $I=0$

Stable oscillation $I=0.2$
FitzHugh-Nagumo Orbits
References